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Adaptive Channel Memory Truncation for Maximum Likelihood Sequence Estimation

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Maximum likelihood data sequence estimation, implemented by a dynamic programming algorithm known as the Viterbi algorithm (VA), is of considerable interest for data transmission in the presence of severe intersymbol interference and additive Gaussian noise. Unfortunately, the required number of receiver operations per data symbol is an exponential function of the duration of the channel impulse response, resulting in unacceptably large receiver complexity for high-speed PAM data transmission on many channels.

We propose a linear prefilter to force the overall impulse response of the channel/prefilter combination to approximate a desired truncated impulse response (DIR) of acceptably short duration. Given the duration of the DIR, the prefilter parameters and the DIR itself can be optimized adaptively to minimize the mean-square error between the output of the prefilter and the desired prefilter output, while constraining the energy in the DIR to be fixed.

In this work we show that the minimum mean-square error can be expressed as the minimum eigenvalue of a certain channel-dependent matrix, and that the corresponding eigenvector represents the optimum DIR. An adaptive algorithm is developed and successfully tested. The simulations also show that the prefiltering scheme, used together with the VA for two different channel models, compares favorably in performance with another recently proposed prefiltering scheme. Limiting results for the case where the prefilter is considered to be of infinite length are obtained; it is shown that the optimum DIR of length two must be one of two possible impulse responses related to the duobinary impulse response. Finally we obtain limiting results for the case where the transmitting filter is optimized.

I. INTRODUCTION

Forney[1] has recently proposed a receiver structure for a communication system operating over a known time-dispersive channel with little loss in performance due to intersymbol interference by using maximum likelihood sequence estimation, or the Viterbi algorithm (VA).[2] This has resulted in much attention being given to practical methods of applying his results. Magee and Proakis[3] proposed the use of the VA directly in conjunction with a channel estimator. This approach can result in a receiver too complex for practical use because the complexity of the VA depends exponentially on the duration of the channel impulse response.

In particular, if the impulse response of the channel has an effective duration of τ seconds and if an L-level PAM system transmits $1/T$ data symbols per second, the number of operations per received symbol is proportional to $L^{\tau/T}$. For channels such as voiceband telephone channels, the bandwidth of which is used efficiently, typical values of τ/T may be between about 20 and 200, making direct application of the VA infeasible.

Thus, it seems clear that effective practical application of the VA or of related techniques involves a compromise between optimum performance and receiver complexity. The complexity-limiting approach we take here is to use a linear prefilter at the receiver to "condition" the overall sampled impulse response seen by the VA so that it is significantly different from zero over only a small number of samples, and any remaining intersymbol interference is considered to be noise. Additional joint optimization of the transmitting filter is also treated, but would be much harder to implement in a real system.

The simplest example of a prefilter is a linear equalizer, which yields an approximate overall impulse response of just one sample. Another example of prefiltering for a different purpose is the linear portion of a decision-feedback equalizer; in that case the initial sample of the desired overall impulse response is required to be large relative to the additive noise.

In any application of prefiltering to approximate a desired impulse response (DIR), the DIR itself and the prefilter should be chosen to minimize the error due to noise and to the difference between the DIR and the actual impulse response that is achieved. The latter error results from intersymbol interference components outside the interval accounted for by the DIR samples as well as from errors in approximating the DIR inside the time-limited interval. This error could be

eliminated hy using a zero-forcing criterion at the cost of additive noise.

Qureshi and Newhall[4] have recently proposed a receiver incorporating prefiltering with the VA. They use a mean-square error (MSE) criterion to force the overall response of the channel plus the linear equalizer to approximate a truncated version of the channel pulse response. In order to deeode, the VA assumes this truncated response, resulting in much simplified processing. There is no effort made in Ref. 4 to optimize the desired truncated response. It is the purpose of this paper to see how this desired response can he chosen to minimize MSE and to show that this receiver structure can he made adaptive.

In Section II we formulate the MSE-minimization prohlem, assuming a fixed numher of samples in the DIR and in the impulse response of the prefilter. The minimum achievahle MSE is the minimum eigenvalue of a certain channel-dependent matrix. In Section III we indicate how the prefilter tap coefficients and the samples of the DIR can he determined adaptively hy a gradient algorithm hased on the MSE minimization. Section IV is a study of the limiting situation in which the tapped delay line prefilter consists of an infinite numher of taps and it is preceded by a matched filter. Compact expressions for the prefilter impulse response, DIR, and minimum MSE are derived, which lend further insight. Section V describes the results of computer simulations of an adaptive prefilter/VA receiver structure, including comparison of the receiver with that descrihed in Ref. 4 and with performance lower hounds. Section VI presents performance ealculations for the prefilter/VA system, a decision-feedhack equalizer, and a linear equalizer for a particular channel. Plots of minimum MSE versus hit rate for each of the three types of receiver structures are shown. Section VII considers asymptotic transmitter optimization.

II. OPTIMIZATION OF THE RECEIVER

The channel model and the preliminary receiver processing are shown in Fig. 1. The channel is modeled as a linear continuous filter with additive white Gaussian noise. It has heen shown hy Forney[1] that the channel can then he followed hy a matched filter, symhol rate sampler, and noise whitening filter with no loss of information. Alternatively, the reader may assume that the channel is handlimited and symhol rate sampling can he used with no information loss.

Due to these considerations, the discrete-time model of Fig. 2 was adopted with the additional assumption that the channel pulse re-

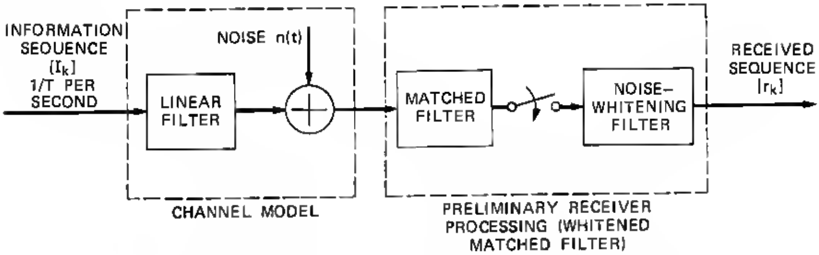

Fig. 1—Channel model and preliminary receiver processing.

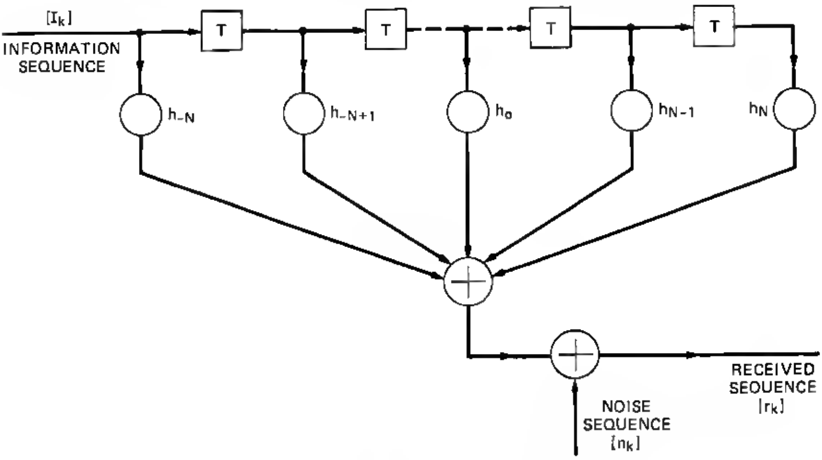

Fig. 2—Discrete-time channel model.

sponse is time-limited. The noise sequence $\{n_k\}$ is additive, uncorrelated, and Gaussian with variance σ^2. Note that a discrete-time model with uncorrelated noise samples also results from the commonly used but nonoptimum expedient of passing the received signal through a flat Nyquist band-limiting filter prior to sampling.*

The proposed receiver structure is shown in Fig. 3. The received sequence feeds a linear tapped delay line filter whose function is to shorten the overall impulse response length. The filter has $L(=2M+1)$ taps which are chosen in the manner to be described later. The output of this filter feeds the Viterbi algorithm which detects the information sequence.

* Although a white noise model was used throughout, the correlated noise case can be considered in a similar manner.

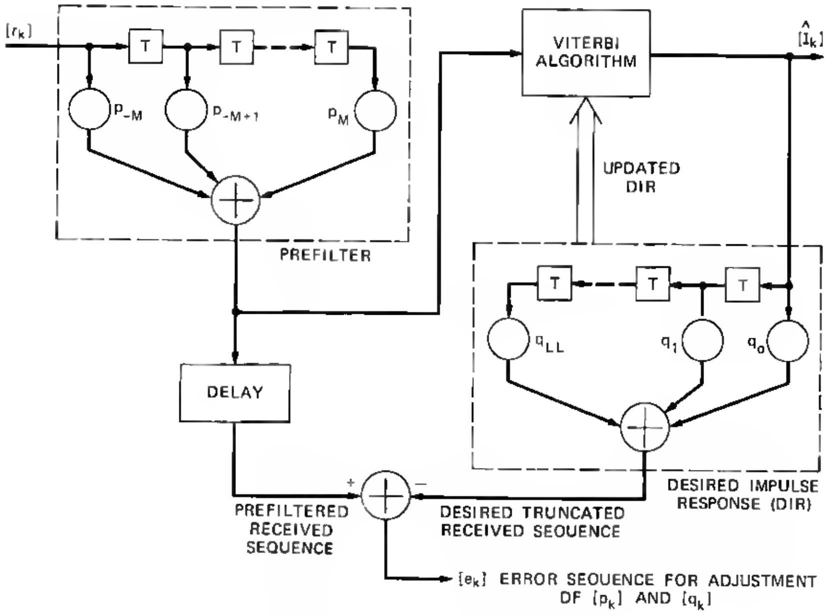

Fig. 3—Receiver structure.

The Viterbi algorithm makes decisions on the assumption that the DIR $\{q_k\}_{k=0}^{LL}$ is the actual overall channel response. The value of LL (the length of the DIR) is much less than $2N + 1$ (the length of the actual channel response). LL is chosen to make acceptable the complexity of the Viterbi algorithm while taking a small noise penalty in the linear preprocessing.* An error signal is formed by feeding the information sequence estimate through the tapped delay line representing the desired channel response. This forms the desired truncated channel received sequence which is then compared with a delayed version of the actual linear prefilter output to form an error sequence. It is this error which is to be minimized since it represents a sum of the additive noise, and the difference between the desired and actual overall impulse responses.

If the sampled channel impulse response, sequence of information symbols, and sequence of uncorrelated noise samples are represented respectively by $\{h_l\}_{l=-\infty}^{\infty}$, $\{I_l\}_{l=-\infty}^{\infty}$, and $\{n_l\}_{l=-\infty}^{\infty}$, then the kth dis-

* Obviously if LL is allowed to be very large, the DIR can closely approximate a delayed version of the original channel impulse response, and there is no significant noise penalty, since the prefilter simply approximates a delay line.

crete channel output is

$$r_k = \sum_l h_l I_{k-l} + n_k. \tag{1}$$

Then if the vectors $\mathbf{P}^+ \equiv (p_{-M}, \cdots, p_0, \cdots p_M)$ where + indicates transpose and $\mathbf{Q}^+ \equiv (q_o, \cdots, q_{LL})$ represent the tap coefficients of the prefilter and the DIR respectively, the error in the kth interval is

$$e_k = \sum_{l=-M}^{M} p_l r_{k-l} - \sum_{l=o}^{LL} q_l I_{k-l}. \tag{2}$$

In order to simplify the following equations, it is assumed that the information sequence is uncorrelated $(\overline{I_k I_j} = \delta_{kj})$ and that the information sequence estimate equals the information sequence. Substituting (1) into (2) and averaging e_k^2 we get

$$\overline{e_k^2} \equiv \overline{e^2} = \mathbf{P}^+ A\mathbf{P} + \mathbf{Q}^+\mathbf{Q} - 2\mathbf{P}^+ H\mathbf{Q}, \tag{3}$$

where

$$H = \begin{bmatrix} h_M & \cdots h_{M+LL} \\ \vdots & \vdots \\ h_0 & \cdots h_{LL} \\ \vdots & \vdots \\ h_{-M} & \cdots h_{-M+LL} \end{bmatrix} \tag{4}$$

is an $L \times (LL+1)$ matrix and A is an $(L \times L)$ channel covariance matrix with elements $a_{ij} = \overline{r_i r_j}$.

First, the error is minimized with respect to the prefilter by taking the gradient with respect to the taps $\{p_i\}$ and setting it equal to zero. The taps $\{q_i\}$ are constrained to be nonzero.

$$\frac{\partial \overline{e^2}}{\partial \mathbf{P}} = 2A\mathbf{P} - 2H\mathbf{Q} = \mathbf{0} \tag{5}$$

and therefore

$$\mathbf{P}_{\text{opt}} = A^{-1} H\mathbf{Q}. \tag{6}$$

The interpretation—thus far—is that some desired (and truncated) channel response is chosen; and the linear prefilter taps are chosen to force the overall response to this with a minimum MSE. The question of what this desired response should be naturally arises. If a fixed length is assumed for this desired response, the desired response can be optimized in the sense of minimizing MSE. Substituting (6) into (3), one obtains

$$\overline{e^2} = \mathbf{Q}^+[I - H^+ A^{-1} H]\mathbf{Q}, \tag{7}$$

where I is the identity matrix.

Since $\overline{e^2} \geqq 0$, this is a positive definite quadratic form in $\mathbf{Q}$ which depends only upon $\mathbf{Q}$ and the channel characteristics. This can be minimized by choosing $\mathbf{Q}$ to be the eigenvector with the minimum eigenvalue of the matrix $[I - H^+A^{-1}H]$. The constraint

$$\mathbf{Q}^+\mathbf{Q} = 1 \tag{8}$$

is necessary to avoid the trivial case of no MSE. The trivial case corresponds, of course, to no transmission through the channel.

It should be noted that when the MSE is minimized a reasonable definition of the signal-to-noise ratio (SNR) seen by the Viterbi algorithm is maximized. This is true because $\{q_i\}$ is considered to be the effective channel pulse response, constrained to unit energy; the additive noise plus any residual intersymbol interference is the effective noise seen by the algorithm. Since this noise is equal to the MSE which has been minimized, the SNR has been maximized.

In summary, to minimize the MSE and thus maximize the SNR seen by the VA receiver, choose

$$\mathbf{Q}_{\text{opt}} = \text{eigenvector of } [I - H^+A^{-1}H] \text{ corresponding to its minimum eigenvalue,} \tag{9}$$

$$\mathbf{P}_{\text{opt}} = A^{-1}H\mathbf{Q}_{\text{opt}}, \tag{10}$$

and then

$$\overline{e^2}\,\text{min} = \text{min eigenvalue of } [I - H^+A^{-1}H]. \tag{11}$$

III. AN ADAPTIVE ALGORITHM FOR OPTIMUM RECEPTION

In order to make the receiver structure practical, the procedure of choosing the $\{p_i\}$ and the $\{q_i\}$ must be made adaptive since the channel pulse response will not usually be known prior to the start of transmission. An algorithm to choose the taps adaptively will now be described.

Consider the conditions for the optimum operating point of this receiver to be reached. The condition that the gradient with respect to $\mathbf{P}$ be equal to zero is easily implemented by using the products of sampled values of quantities in the receiver as noisy estimates of the required cross correlations, assuming the data sequence is known or has been correctly estimated by the receiver. Thus,

$$\mathbf{P}^{(r+1)} = \mathbf{P}^{(r)} - \Delta_1 e_r \mathbf{R}^{(r)}, \tag{12}$$

where $\mathbf{P}^{(r)}$ is the set of tap values at the rth iteration, Δ_1 is an adjustment parameter which controls accuracy and speed of convergence,

$\mathbf{R}^{(r)}$ is a vector of the received samples contained in the linear preprocessing filter, and e_r is the error in (2). $\mathbf{P}^{(r+1)}$ is thus the new estimate of the $\{p_i\}$ taps, and when a steady state is reached a noisy unbiased estimate of these taps is obtained. Note that the value of $\mathbf{P}$ implicitly depends on the value of $\mathbf{Q}$ through the e_r terms.

The algorithm to obtain the $\mathbf{Q}$ taps is not so easily obtained. Consider the unconstrained gradient with respect to the $\mathbf{Q}$ vector. If a noisy estimate of the required cross correlation is used, then the recursion for the unconstrained gradient algorithm is

$$\mathbf{Q}^{(r+1)} = \mathbf{Q}^{(r)} + \Delta_2 e_r \mathbf{I}^{(r)}, \tag{13}$$

where $\mathbf{I}^{(r)}$ is a vector of the information symbols contained in the channel reference filter. If (12) and (13) are followed without a constraint at each iteration, then the trivial solution results. The algorithm is therefore modified so that the $\mathbf{Q}$ vector is renormalized at each step. That is,

$$\tilde{\mathbf{Q}}^{(r+1)} = \mathbf{Q}^{(r)} + \Delta_2 e_r \mathbf{I}^{(r)} \tag{14}$$

$$\mathbf{Q}^{(r+1)} = \frac{\tilde{\mathbf{Q}}^{(r+1)}}{(\tilde{\mathbf{Q}}^{(r+1)})^{+}(\tilde{\mathbf{Q}}^{(r+1)})}. \tag{15}$$

By following the combined algorithm of (12), (14), and (15), a stationary point in $\mathbf{P}$ will be reached, and the energy in $\mathbf{Q}$ will be constrained to one.

Now consider the noiseless unconstrained gradient of $\overline{e^2}$ with respect to $\mathbf{Q}$. Then

$$\frac{\partial \overline{e^2}}{\partial \mathbf{Q}} = 2\mathbf{Q} - 2H^{+}\mathbf{P}. \tag{16}$$

Consider $\mathbf{P}$ to be in the neighborhood of the correct solution (6) with respect to $\mathbf{Q}$ (that is, $\mathbf{P}$ is adjusted more quickly than $\mathbf{Q}$). Then (16) becomes

$$\frac{\partial \overline{e^2}}{\partial \mathbf{Q}} = 2\mathbf{Q} - 2H^{+}A^{-1}H\mathbf{Q}. \tag{17}$$

Thus the gradient algorithm, in terms of the actual matrix quantities, becomes

$$\begin{aligned}\tilde{\mathbf{Q}}^{(r+1)} &= \mathbf{Q}^{(r)} - \tfrac{1}{2}\Delta_2 \frac{\partial \overline{e^2}}{\partial \mathbf{Q}^{(r)}} \\ &= \mathbf{Q}^{(r)} - \tfrac{1}{2}\Delta_2(2\mathbf{Q}^{(r)} - 2H^{+}A^{-1}H\mathbf{Q}^{(r)}) \\ &= \Delta_2 H^{+}A^{-1}H\mathbf{Q}^{(r)} + \mathbf{Q}^{(r)}(1 - \Delta_2)\end{aligned} \tag{18}$$

and then $\tilde{\mathbf{Q}}^{(r+1)}$ is renormalized to form $\mathbf{Q}^{(r+1)}$. Now note that if

$\Delta_2 = 1$ this corresponds exactly to the method of Vianello and Stodola[5] for determining the maximum eigenvalue and corresponding eigenvector of $H^+A^{-1}H$. Since the maximum eigenvalue of $H^+A^{-1}H$ corresponds to the minimum eigenvalue of $(I - H^+A^{-1}H)$ this technique will converge to the minimum MSE. This method will fail only when the starting vector $\mathbf{Q}^{(1)}$ is exactly orthogonal to the desired solution. Since the algorithm actually used (14)–(15) deals with noisy estimates rather than the exact expressions, the noise will prevent the case of the algorithm becoming stuck on a veetor orthogonal to the solution.

In the practical case it is not possihle to choose Δ_2 to he one hecause when the noisy estimates are used the algorithm will amplify the noise and diverge. Actually, Δ_2 will be much smaller than one. Again looking at (18), one can see that a steady state is reached when $\mathbf{Q}$ becomes nonrotating with respect to the transformation. This occurs when $\mathbf{Q}$ is the maximum eigenvalue of $H^+A^{-1}H$ (i.e., the maximum eigenvalue will dominate as in the method of Vianello and Stodola). Thus, the unique solution for $\mathbf{Q}$ has heen ohtained.

IV. LIMITING RESULTS

We now study the limiting situation where the prefilter is allowed to be any general linear filter with impulse response $p(t)$, while the desired impulse response $\{q_m\}_{m=0}^{LL}$ is still finite. In addition we assume that the additive noise on the channel is white, with double-sided power spectral density $N_o/2$.

In this case we wish to minimize the mean square of the sampled error

$$e_k = \int_{-\infty}^{\infty} p(\tau)r(kT - \tau)d\tau - \sum_{l=o}^{LL} q_l I_{k-l}, \tag{19}$$

where

$$r(kT - \tau) = \sum_{l=-\infty}^{\infty} h(kT - lT - \tau)I_l + n(kT - \tau) \tag{20}$$

is the received signal.

The MSE is then

$$\overline{e^2} = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\tau_1)p(\tau_2)h(lT - \tau_1)h(lT - \tau_2)d\tau_1 d\tau_2 + \frac{N_o}{2}\int_{-\infty}^{\infty} p(\tau)^2 d\tau - 2\sum_{l=o}^{LL} q_l \int_{-\infty}^{\infty} p(\tau)h(lT - \tau)d\tau + \sum_{l=o}^{LL} q_l^2. \tag{21}$$

Using a simple calculus of variations argument to minimize MSE with respect to the prefilter impulse response $p(t)$, we get the following integral equation defining the optimum $p(t)$.

$$\frac{N_o}{2} p(t) = \sum_{l=o}^{LL} q_l h(lT - t) - \sum_{l=-\infty}^{\infty} S_l h(lT - t) , \tag{22}$$

where

$$S_l = \int_{-\infty}^{\infty} p(\tau) h(lT - \tau) d\tau \tag{23}$$

is the overall sampled impulse response of the channel and prefilter. Note that we would hope for $0 \leqq l \leqq LL$, $S_l \approx q_l$ and for $l < 0$ and $l > LL$, $S_l \approx 0$.

Equation (22) tells us that the optimum prefilter structure is a matched filter with impulse response $h(-t)$, followed by an infinite-length tapped delay line whose tap gains $\{p_l\}$ are given by (22) and (23).

$$\frac{N_o}{2} p_l = q_l - \sum_{l=-\infty}^{\infty} p_l \phi_{m-l} , \tag{24a}$$

where

$$\phi_m = \int_{-\infty}^{\infty} h(mT - \tau) h(-\tau) d\tau = \phi_{-m} \tag{24b}$$

is the channel's sampled covariance function, and where we later require that $\{q_l\}$ is nonzero only for $0 \leqq l \leqq LL$. We remark that the development so far is analogous to that of Berger and Tufts[6] for the case $LL = 0$. Equation (24) may be solved in terms of z-transforms. Defining

$$q(z) \equiv \sum_{m=-\infty}^{\infty} q_m z^m$$

$$p(z) = \sum_{m=-\infty}^{\infty} p_m z^m$$

$$\phi(z) = \sum_{m=-\infty}^{\infty} \phi_m z^m$$

we can take z-transforms of both sides of (24) and solve for $p(z)$.

$$p(z) = \frac{q(z)}{\phi(z) + \frac{N_o}{2}} , \tag{25}$$

where we have used the fact that $\phi(z) = \phi(z^{-1})$ since the sequence $\{\phi_l\}$ is symmetric about $l = o$.

Using (25) we get the z-transform of the autocovariance sequence of the $\{e_k\}$, when the tap coefficients $\{p_n\}_{-\infty}^{\infty}$ are chosen to minimize the mean-squared error. Defining $E_m = \overline{e_k e_{k+m}}$ and $E(z) = \sum_{m=-\infty}^{\infty} E_m z^m$ we have

$$E(z) = \frac{N_o}{2} \frac{q(z)q(z^{-1})}{\phi(z) + \frac{N_o}{2}}. \tag{26}$$

We now minimize $\overline{e^2} = E_o$ with respect to the desired impulse response samples $\{q_n\}_{n=o}^{LL}$, under an appropriate energy constraint. Taking the inverse transform of $E(z)$ we have

$$\begin{aligned} E_o &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} E(e^{-j\omega T}) d\omega \\ &= \frac{N_o T}{4\pi} \int_{-\pi/T}^{\pi/T} \frac{|q(e^{j\omega T})|^2}{\phi(e^{-j\omega T}) + \frac{N_o}{2}} d\omega\,, \end{aligned} \tag{27}$$

where

$$q(e^{j\omega T}) = q_o + q_1 e^{j\omega T} + \cdots + q_{LL} e^{j\omega LLT}.$$

Defining the $LL + 1$ dimensional vector $\mathbf{Q}^+ = (q_o,\ q_1, \cdots q_{LL})$ we can rewrite (27) as a quadratic form

$$E_o = \mathbf{Q}^+ R \mathbf{Q}, \tag{28}$$

where R is a square matrix of dimension $(LL + 1)$ whose i–jth element is

$$r_{ij} = \frac{N_o T}{4\pi} \int_{-\pi/T}^{\pi/T} \frac{e^{j\omega(i-j)T}}{\phi(e^{j\omega T}) + \frac{N_o}{2}} d\omega \cdot \tag{29}$$

Note that $\phi(e^{j\omega T})$ is the discrete Fourier transform of an autocovariance sequence, and hence is an even, real, positive function of ω. Thus $r_{ij} = r_{ji}$ is a real function of $|i - j|$, and so R is a positive definite symmetric Toeplitz matrix.

Minimization of E_o under the energy constraint $|\mathbf{Q}|^2 = 1$ is then accomplished by making $\mathbf{Q}$ that normalized eigenvector of R corresponding to its minimum eigenvalue. The matrix R is evidently the limiting case of the matrix $I - H^+A^{-1}H$ for the finite-tap receiver [displayed in expressions (7) through (11)]. Then

$$\overline{e^2_{\min}} = \lambda_{\min}(R). \tag{30}$$

To recapitulate, the minimum is taken over the set of tap-coefficients

$\{p_n\}_{n=-\infty}^{\infty}$ and $\{q_n\}_{n=o}^{LL}$ under the constraint $\sum_{n=o}^{LL} q_n^2 = 1$,

which can he expressed as

$$\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}|q(e^{j\omega T})|^2 d\omega = 1.$$

Thus, from (27) we have the lower bound

$$\lambda_{\min}(R) = E_o = \overline{e^2_{\min}} \geqq \frac{N_o}{2}\frac{1}{\sup\limits_{-\pi/T \leqq \omega \leqq \pi/T}\left(\phi(e^{j\omega T}) + \frac{N_o}{2}\right)}. \tag{31}$$

Now $\phi(e^{j\omega T})$ is the discrete Fourier transform of the sequence $\{\phi_n\}$ defined in terms of the channel's impulse response hy (24h). Thus, if the channel's transfer function is denoted hy

$$H(\omega) = \int_{\infty}^{\infty} h(t)e^{-j\omega T}dt\,,$$

$$\phi(e^{j\omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\left|H\left(\omega + \frac{2n\pi}{T}\right)\right|^2. \tag{32}$$

The term $\phi(e^{j\omega T})$ can be interpreted as the channel's "folded" power spectrum.[7]

When $LL + 1$, the numher of components in the desired impulse response $\{q_n\}_{n=o}^{LL}$, is relatively small, say less than 10, the minimum eigenvalue and corresponding eigenvector of R can he evaluated without difficulty. For much longer values of LL, the lower bound (31) which is easily computed using (32) may he quite tight. A particular case of interest is where $LL = 1$. Then R has the form

$$R = \begin{bmatrix} r_o & r_1 \\ r_1 & r_o \end{bmatrix}$$

and

$$\overline{e^2_{\min}} = \lambda_{\min}(R) = \min(r_o + r_1, r_o - r_1),$$

where

$$r_o + r_1 = \frac{N_o T}{2\pi}\int_{-\pi/T}^{\pi/T}\frac{\cos^2\frac{\omega T}{2}}{\phi(e^{j\omega T}) + \frac{N_o}{2}}d\omega \tag{33}$$

and

$$r_o - r_1 = \frac{N_o T}{2\pi}\int_{-\pi/T}^{\pi/T}\frac{\sin^2\frac{\omega T}{2}}{\phi(e^{j\omega T}) + \frac{N_o}{2}}d\omega\cdot \tag{34}$$

The normalized eigenvectors (optimum (q_o,q_1)) corresponding to the eigenvalues $r_o + r_1$ and $r_o - r_1$ are respectively $(1/\sqrt{2},\ 1/\sqrt{2})$ and $(1/\sqrt{2},\ -1/\sqrt{2})$.

Thus, we have the curious result that the optimum desired impulse response of length two is one of only two possihle forms, depending only on whether the channel's folded power spectrum is such that (33) or (34) is smaller. For example, if the channel's folded power spectrum has a single minimum near the hand edge, $\omega = \pi/T$, the best choice for (q_o,q_1) would be $(1/\sqrt{2},\ 1/\sqrt{2})$ since $\cos^2 (\omega T)/2$ has a zero at the hand edge. However, if the channel's folded power spectrum has a single minimum near zero frequency, the hest choice for (q_o,q_1) would he $(1/\sqrt{2},\ -1/\sqrt{2})$, since $\sin^2 (\omega T)/2$ is zero at $\omega = o$. These two cases are illustrated in Fig. 4.

It is interesting to point out that the two possihle optimum desired impulse responses $(1/\sqrt{2},\ 1/\sqrt{2})$ and $(1/\sqrt{2},\ -1/\sqrt{2})$ are reminiscent of duohinary and partial response impulse responses.[8]

V. PERFORMANCE OVER SIMULATED CHANNELS

In order to ohserve performance ohtainahle from this receiver structure, the arhitrary discrete time channels shown in Fig. 5 were used. Figure 6 shows the results of the simulations performed with the receiver developed here and that of Qureshi and Newhall on these channels. Underlined in Fig. 5 are the desired response used for the

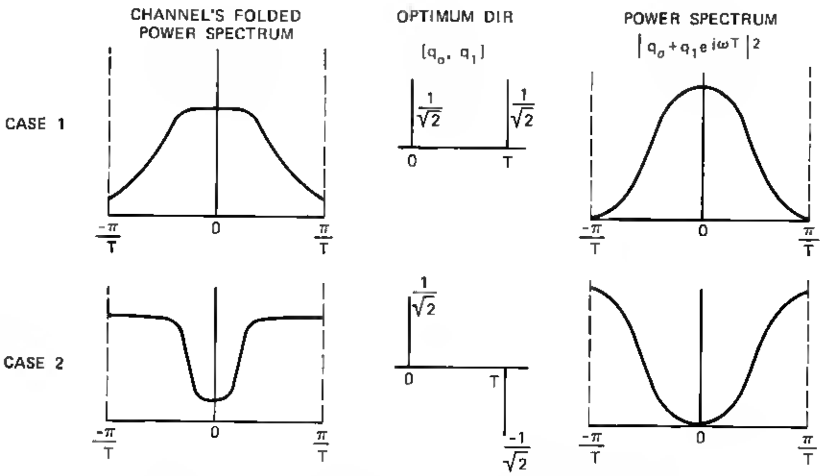

Fig. 4—Optimum desired impulse response of length two.

-0.1, -0.2, -0.1, 0.3, -0.3, 0.33, 0.67, 1.0, 0.67, 0.33, 0.3, -0.5, -0.1, 0.1, 0.1

CHANNEL A

-0.1, -0.1, -0.3, -0.15, -0.33, 0.16, 0.33, 1.0, 0.33, 0.16, -0.33, -0.3, -0.15, -0.1, -0.1

CHANNEL B

Fig. 5—Sampled channel impulse responses used in the simulation.

Qureshi and Newhall receiver. As can be seen from the performance curves, the Qureshi and Newhall receiver performs about as well as our receiver for Channel A and much worse for Channel B. The difference in performance is presumably due to the use of different criteria to choose the desired response. In the case of Channel B, the channel passes virtually no dc, yet the DIR from truncating the channel response does pass dc. This causes considerable noise enhancement by the Qureshi and Newhall linear prefilter.

Figure 6 also shows the matched filter lower bound, the lower bound on performance derived by Forney,[9] and a lower estimate which is used to predict actual optimum reference receiver performance. This lower estimate is obtained by computing the MSE and minimum coding distance of the DIR. Thus, it is assumed that the MSE is uncorrelated and Gaussian in this approximation.

$$P(e) \lesssim K \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{d_{\min}^2}{2\,\mathrm{MSE}}}\right), \tag{35}$$

where K is a constant depending on the error structure of the channel, and $d_{\min}$ is the minimum Euclidian distance between all possible pairs of noiseless sequences with differing first information symbols emerging from the prefilter.[9] This lower estimate is found without considering the fact that the noise is correlated. If a more accurate estimate of performance is desired, the results of Qureshi and Newhall[4] can be used to consider the effects of noise correlation.

The simulations were run with a 31-tap prefilter whose taps were adjusted with $\Delta = 0.001$, and a 5-tap desired overall response length with the adjustment parameter equal to 0.01. In the case of the Qureshi and Newhall receiver, the prefilter was adjusted with $\Delta = 0.001$ and the channel was estimated with a filter with adjustment parameter equal to 0.01.* As the curves show, the receiver structure given here

* The performance loss due to the adjustment parameters has not been evaluated; however, simulation results indicate that this loss is very small.

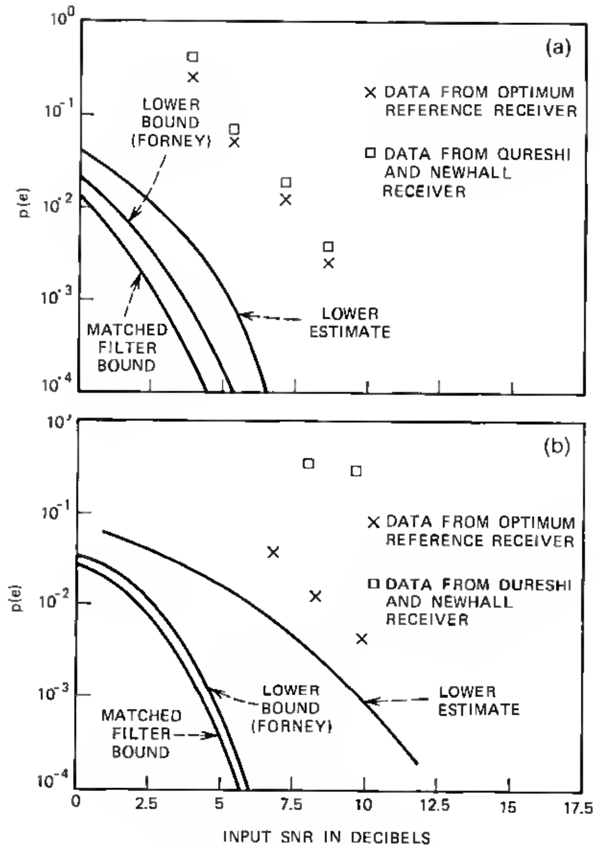

Fig. 6—Simulation results: (a) channel A; (b) channel B.

assures that a good choice of a DIR is made which is not always the same as the truncated channel impulse response.

VI. COMPARISON WITH OTHER SYSTEMS—AN EXAMPLE

Based on the results in Section II, performance calculations were made for baseband PAM transmission on the channel whose frequency response is shown in Fig. 7. The results shown in Fig. 8 were made with the following assumptions:

(*i*) A matched filter preceded the receiver.
(*ii*) There was a 31-tap prefilter.
(*iii*) There was a 5-tap desired impulse response.

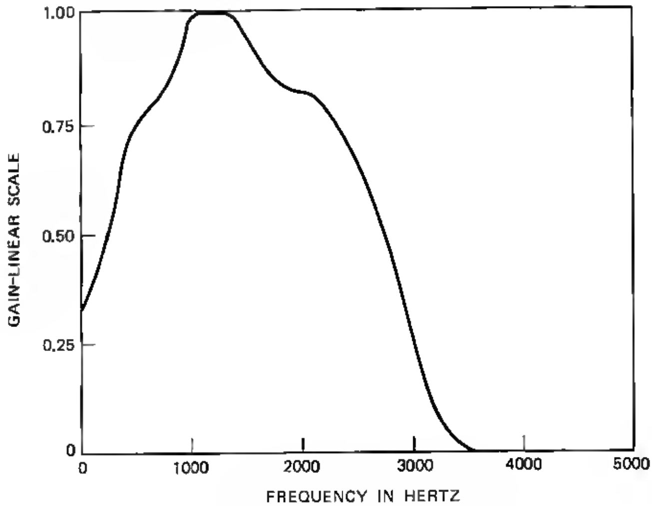

Fig. 7—Channel amplitude characteristic.

(*iv*) Although the noise may be correlated, at tbe input to tbe Viterbi algorithm, it has a negligible effect on performance.

Of the assumptions made, only the one about the noise correlation might not be realistic. Work is currently being done to deal witb tbe correlated noise problem. In any case, it is not expected that it would affect performance more than a few dB and it clearly would not affect the place in the performance curve at which the performance begins to degrade seriously.

Tbe curves representing the linear and decision-feedback equalizers, provided by J. Salz, show the MSE versus rate for additive white Gaussian noise with $N_o/2 = 0.0001$. In the linear and decision-feedback cases the MSE may be roughly related to performance in terms of probability of error.[*6,10] The curve for the prefilter/VA combination, labeled "VA equalizer," is a plot of $(\text{MSE}/d^2_{\min})$ versus rate, wbere $d_{\min}$ is the minimum distance for the DIR. This is done because tbe attainable system performance is not determined by MSE alone, but ratber by $\text{MSE}/d^2_{\min}$ as in expression (35). Direct minimization of this ratio by analytical or numerical means has not been accomplished. Note bowever that tbe minimum value $d_{\min}$ can attain (over all

* The analysis for decision-feedback equalization ignores the effect of decision errors on the MSE.

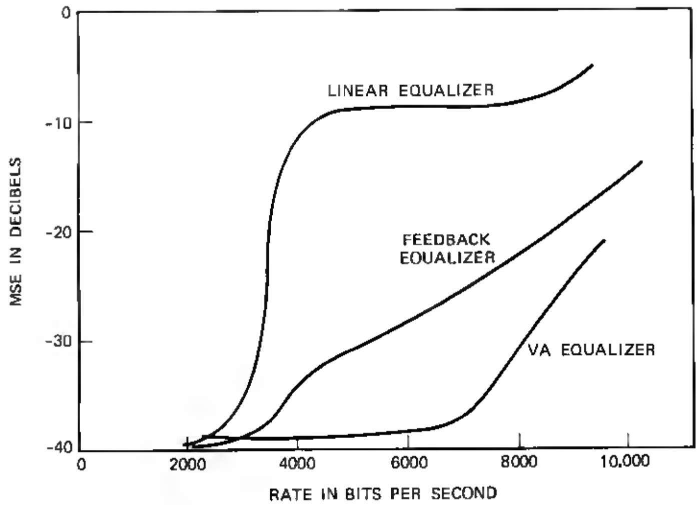

Fig. 8—Indication of attainable performance for three receiver structures.

channels with equal-energy impulse responses) is limited by the duration of the DIR which is chosen.[11]

As can be seen from the curves, this receiver structure can he expected to perform well while using only binary signaling over a much greater range of transmitted data rates than the linear and decision-feedback receivers. This result occurred despite the fact that the linear and decision-feedback computations were made for infinite filters while the prefilter was finite. It is the more relaxed criterion for our system compared to the decision-feedhack criterion which results in lower MSE and thus better performance. Nevertheless the results are considered preliminary until a better understanding of the effect of noise correlation is achieved.

VII. TRANSMITTER OPTIMIZATION

The "channel's" frequency response $H(\omega)$ actually includes the transmitting filter, i.e.,

$$H(\omega) = C(\omega)G(\omega), \tag{36}$$

where $C(\omega)$ is the frequency response of the transmission channel alone, and $G(\omega)$ is the frequency response of the transmitting filter, which we have hitherto assumed fixed. In a practical data communication system, a "reasonable" transmitting filter would likely be fixed to avoid having to provide an extra feedback channel for adjust-

ing the transmitter parameters, and because of the complexity of the transmitter optimization argument itself for general channels.[6]

Nevertheless, the performance attainable with transmitter optimization is of theoretical interest. In this section we obtain expressions for the optimum transmitter filter $G(\omega)$ and the resulting minimum MSE under a transmitted power constraint. For simplicity, we assume a "well behaved" channel $C(\omega)$ for which $|C(\omega)|$ is monotone decreasing, and for which $|C(\omega)|/N_o$ is sufficiently large in the range $\{-\pi/T, \pi/T\}$ that the optimum transmitter uses the entire Nyquist band $|\omega| \leqq \pi/T$. Treatment of more general channel characteristics is more complicated, but can be carried out as in Ref. 6.

The minimum MSE for a fixed transmitting filter $G(\omega)$ and DIR $\{q_l\}$ is given by eq. (27) and by the channel's folded power spectrum, which from (32) and (36) can be written

$$\phi(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left| C\left(\omega - \frac{2\pi n}{T}\right) \right|^2 \left| G\left(\omega - \frac{2\pi n}{T}\right) \right|^2. \tag{37}$$

The constraint that the transmitted power be fixed at P_T can be written

$$\frac{1}{T} \int_{-\pi/T}^{\pi/T} \sum_n \left| G\left(\omega - \frac{2\pi n}{T}\right) \right|^2 d\omega = P_T. \tag{38}$$

A necessary condition for minimizing the MSE, given by (27), subject to the power constraint (38), is obtained using a simple variational argument: for $-\pi/T \leqq \omega \leqq \pi/T$ and every integer m, either $G(\omega - 2\pi m/T) = 0$ or $G(\omega - 2\pi m/T) \neq 0$ and

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} \left| C\left(\omega - \frac{2\pi n}{T}\right) \right|^2 \left| G\left(\omega - \frac{2\pi n}{T}\right) \right|^2 + \frac{N_o}{2} = \lambda |q(e^{j\omega T})| \left| C\left(\omega - \frac{2\pi m}{T}\right) \right|, \tag{39}$$

where λ is a Lagrange multiplier whose value will be determined from the constraint (38). Furthermore, for any ω such that $C(\omega) = 0$, $G(\omega) = 0$.

For any frequency ω, there will be only one integer m for which $G(\omega - 2\pi m/T) \neq 0$, since the left-hand side of (39) does not depend on m and the right-hand side does. Indeed, if $|C(\omega)|$ is monotone decreasing, then best use is made of the transmitter power if $G(\omega) = 0$ for $|\omega| > \pi/T$. Thus we can rewrite (39) as

$$\frac{1}{T}|C(\omega)|^2 |G(\omega)|^2 + \frac{N_o}{2} = \lambda |q(e^{j\omega T})| |C(\omega)| \quad \text{for } C(\omega) \neq 0 \text{ and } |\omega| \leqq \pi. \tag{40}$$

For simplicity, we assume that P_T and the ratio $|C(\omega)|/N_o$ are sufficiently large that (40) can be satisfied for all $|\omega| < \pi/T$. [Otherwise $G(\omega)$ would be zero[6] beyond a certain frequency $\omega_0 < \pi/T$.] Then the amplitude frequency response of the optimum transmitting filter is given by

$$\begin{aligned} \frac{1}{T}|G_{\text{opt}}(\omega)|^2 &= \frac{\lambda|q(e^{j\pi T})|}{|C(\omega)|} - \frac{N_o}{2|C(\omega)|^2} \qquad \text{for } |\omega| \leqq \frac{\pi}{T} \\ &= 0 \qquad \text{for } \omega > \frac{\pi}{T}. \end{aligned} \tag{41}$$

The Lagrange multiplier λ is determined by (41) and the power constraint (38). Substitution of the expression for $|G_{\text{opt}}|$ into expression (27) for the MSE gives

$$\text{MSE} = \frac{N_o T}{4\pi\lambda} \int_{-\pi/T}^{\pi/T} \frac{|q(e^{j\omega T})|}{|C(\omega)|}\, d\omega\,, \tag{41a}$$

where

$$\lambda = \frac{P_T + \dfrac{N_o}{2} \displaystyle\int_{-\pi/T}^{\pi/T} \frac{1}{|C(\omega)|^2}\, d\omega}{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{|q(e^{j\omega T})|}{|C(\omega)|}}. \tag{41b}$$

It is interesting to look now at the frequency response of the optimum receiver prefilter

$$\frac{1}{T}\,|P_R(\omega)|^2 \equiv \frac{1}{T}\,|C(\omega)|^2\,|G_{\text{opt}}(\omega)|^2\,|p(e^{j\omega T})|^2\,, \tag{42}$$

corresponding to the optimum transmitter filter. The left side of (42) follows from the cascade of the channel and transmitter and the appropriate matched filter, followed by the discrete filter $\{p_l\}$. Substitution of expressions (41) for $|G_{\text{opt}}(\omega)|$ and (25) for $p(e^{j\omega T})$ results in

$$\frac{1}{T}\,|P_R(\omega)|^2 = \frac{1}{\lambda^2 T}\left[\frac{\lambda|q(e^{j\omega T})|}{|C(\omega)|} - \frac{N_o}{2|C(\omega)|^2}\right] \qquad \text{for } |\omega| \leqq \frac{\pi}{T}. \tag{43}$$

Thus the transmitting filter and receiving prefilter frequency responses are identical in the Nyquist band except for constant factors (clearly, the transmitting and receiving filters' phase characteristics can be chosen arbitrarily). This equal sharing of the filtering load between the transmitter and receiver is a well-known result for optimum *linear* communication systems (see pp. 118–121 of Ref. 7).

It is also of interest to evaluate the power spectrum of the error sequence that the Viterbi algorithm assumes to be additive uncorre-

lated Gaussian noise samples. From expressions (26) and (40), this is given hy

$$
\begin{aligned}
E(e^{j\omega T}) &= \frac{N_o}{2} \frac{|q(e^{j\omega T})|^2}{\lambda |q(e^{j\omega T})| \, |C(\omega)|} \\
&= \frac{N_o}{2\lambda} \frac{|q(e^{j\omega T})}{|C(\omega)|} \qquad \text{for } |\omega| \leqq \frac{\pi}{T}. \qquad (44)
\end{aligned}
$$

Thus, the extent that the amplitude frequency response of the chosen DIR approximates that of the channel in the Nyquist band determines how close the power spectrum $E(e^{j\omega T})$ is to heing flat, and hence, to what extent successive errors are uncorrelated.

From (41a) and (41h) we ohtain an expression for the MSE for a given DIR after the transmitting and receiving filters have been optimized.

$$
\text{MSE} = \frac{\frac{N_o T}{4\pi} \alpha^2(\mathbf{Q})}{P_T + \frac{N_o}{2} \int_{-\pi/T}^{\pi/T} \frac{1}{|C(\omega)|^2} d\omega}, \qquad (45a)
$$

where

$$
\begin{aligned}
\alpha(\mathbf{Q}) &= \int_{-\pi/T}^{\pi/T} \frac{|q(e^{j\omega T})|}{|C(\omega)|} d\omega \\
&= \int_{-\pi/T}^{\pi/T} \frac{\left[\sum_{l=0}^{LL} \sum_{m=0}^{LL} q_l q_m e^{j(m-l)\omega T} \right]^{\frac{1}{2}}}{|C(\omega)|} d\omega \cdot \qquad (45b)
\end{aligned}
$$

Minimization of the MSE expression with respect to the DIR $\mathbf{Q}$ under the constraint $|\mathbf{Q}|^2 = 1$ is then equivalent to minimization of $\alpha(\mathbf{Q})$ under this constraint. Necessary conditions for the optimum $\mathbf{Q}^+ = (q_0, \cdots q_{LL})$ are then

$$
\mu q_l = \sum_{m=0}^{LL} q_m \rho_{l-m}(\mathbf{Q}), \qquad l = 0, \cdots LL, \qquad (46a)
$$

where

$$
\rho_l(\mathbf{Q}) = \int_{-\pi/T}^{\pi/T} \frac{e^{jl\omega T}}{|C(\omega)| \left[\sum_{i=0}^{LL} \sum_{k=0}^{LL} q_i q_k e^{j(i-k)\omega T} \right]^{\frac{1}{2}}} d\omega \qquad (46h)
$$

and where μ is a Lagrange multiplier. Again the optimum DIR $\mathbf{Q}$ is the solution of an eigenvalue problem, this time nonlinear. It is easy to verify that the optimum DIR of length two is again either of the two "duobinary" impulse responses shown in Fig. 4. In that case the

minimum achievable MSE is given by (45a) and (45b) as

$$\mathrm{MSE} = \frac{\dfrac{N_oT}{4\pi}\left[\displaystyle\int_{-\pi/T}^{\pi/T} \frac{[1 \pm \cos\omega T]^{\frac{1}{2}}}{|C(\omega)|} d\omega\right]^2}{P_T + \dfrac{N_o}{2}\displaystyle\int_{-\pi/T}^{\pi/T} \frac{1}{|C(\omega)|^2} d\omega}, \tag{47}$$

the $+$ or $-$ being chosen to minimize (47).

VIII. CONCLUSIONS

We have presented a scheme of linear prefiltering to optimally "condition" the impulse response of a channel to approximate an impulse response of limited duration for which maximum likelihood estimation of the data sequence is implementable in practice. This scheme in conjunction with the VA can be adaptive, to deal with unknown or slowly time-varying channels. In the simulations its performance compared favorably with the similarly motivated scheme of Ref. 4.

The optimization criterion we used—minimization of the MSE with respect to the prefilter taps and the DIR, with the energy, duration, and relative delay of the DIR being fixed—is admittedly somewhat *ad hoc*. If the sequence of errors $\{e_i\}$ emerging from the prefilter is still assumed to be stationary Gaussian, with zero mean and covariance $\{E_m\}$, then it can be shown that the error rate of a VA which assumes correlated noise is minimized if a certain weighted minimum distance is maximized, namely

$$\min_{\mathbf{d}\in s'} \mathbf{d}^+\Lambda^{-1}\mathbf{d},$$

where the s' is the set of all possible vectors representing error events and Λ is a covariance matrix whose dimension equals that of $\mathbf{d}\{\Lambda_{ij} = E_{|i-j|}\}$. The above quantity is clearly difficult to maximize, and even if it could be done, the non-Gaussianness of the error sequence would render the solution suspect.

Nevertheless, the performance estimates for the sample channel reported in Section VI make the use of the VA is conjunction with prefiltering appear attractive for high-speed data transmission relative to other schemes. Further studies should be done on the correlatedness of the error sequence and the minimum distance properties of the desired impulse responses.*

* S. Fredricsson presented a paper dealing with this subject at the International Symposium on Information Theory, Israel, June 1973.

IX. ACKNOWLEDGMENTS

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